

Omni-Lie 2-algebras and their Dirac structures *

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Abstract

We introduce the notion of omni-Lie 2-algebra, which is a categorification of Weinstein’s omni-Lie algebras. We prove that there is a one-to-one correspondence between strict Lie 2-algebra structures on 2-sub-vector spaces of a 2-vector space \mathbb{V} and Dirac structures on the omni-Lie 2-algebra $\mathfrak{gl}(\mathbb{V}) \oplus \mathbb{V}$. In particular, strict Lie 2-algebra structures on \mathbb{V} itself one-to-one correspond to Dirac structures of the form of graphs. Finally, we introduce the notion of twisted omni-Lie 2-algebra to describe (non-strict) Lie 2-algebra structures. Dirac structures of a twisted omni-Lie 2-algebra correspond to certain (non-strict) Lie 2-algebra structures, which include string Lie 2-algebra structures.

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1 Introduction

The notion of omni-Lie algebra was introduced by Weinstein in [19] to characterize Lie algebra structures on a vector space V . An omni-Lie algebra can be regarded as the linearization of the Courant algebroid [9, 11] structure on $TM \oplus T^*M$ at a point, where M is a finite dimensional differential manifold. It is studied from several aspects recently [3, 6, 15, 16, 18]. An omni-Lie algebra associated to a vector space V is the direct sum space $\mathfrak{gl}(V) \oplus V$ together with the nondegenerate symmetric pairing $\langle \cdot, \cdot \rangle$ and the skew-symmetric bracket operation $[\![\cdot, \cdot]\!]$ given by

$$\langle A + u, B + v \rangle = \frac{1}{2}(Av + Bu),$$

and

$$[\![A + u, B + v]\!] = [A, B] + \frac{1}{2}(Av - Bu).$$

With the factor of $\frac{1}{2}$, the bracket $[\![\cdot, \cdot]\!]$ does not satisfy the Jacobi identity. However, this bracket can be completed to a structure of a Lie 2-algebra as in [16]. Moreover, this Lie 2-algebra is integrated to a Lie 2-group in [15]. Thus this integration procedure provides another solution to the integration problem of omni-Lie algebras, studied by Kinyon and Weinstein in [6].

Notice that even though the motivating example of Courant bracket involves an infinite dimensional vector space $\chi(M) \oplus \Omega^1(M)$, Weinstein's linearization makes it possible to study a finite dimensional model, namely $\mathfrak{gl}(V) \oplus V$, where V , as the tangent space $T_m M$ at certain point $m \in M$, is a finite dimensional vector space. Thus in our paper, we also restrict ourselves to the finite dimensional case. That is, all the vector spaces in this paper are finite dimensional.

In [3], the authors introduced the notion of omni-Lie algebroid to characterize Lie algebroid structures on a vector bundle E . Omni-Lie algebroids are generalizations of Weinstein's omni-Lie algebras from vector spaces to vector bundles. An omni-Lie algebroid is the direct sum bundle $\mathfrak{D}E \oplus \mathfrak{J}E$ together with an E -valued pairing and a bracket operation, where $\mathfrak{D}E$ and $\mathfrak{J}E$ are the covariant differential operator bundle and the first jet bundle of E respectively. The main result is that Lie algebroid structures on E one-to-one correspond to Dirac structures of the form of graphs. Moreover, (general) Dirac structures one-to-one corresponds to projective Lie algebroid structures on sub-vector bundles of $E \oplus TM$ [4].

Recently, people have payed more attention to higher categorical structures with motivations from string theory. One way to provide higher categorical structures is by categorifying existing mathematical concepts. One of the simplest higher structure is a 2-vector space, which is a categorified vector space. If we further put Lie algebra structures on 2-vector spaces, then we obtain the notion of Lie 2-algebras [1]. The Jacobi identity is replaced by a natural transformation, called Jacobiator, which also satisfies some coherence laws of its own. One of the motivating examples is the differentiation of Witten's string Lie 2-group $String(n)$, which is called a string Lie 2-algebra. As $SO(n)$ is the connected part of $O(n)$ and $Spin(n)$ is the simply connected cover of $SO(n)$, $String(n)$ is a “cover” of $Spin(n)$ which has trivial π_3 (notice that $\pi_2(G) = 0$ for any Lie group

G). The differentiation of $\text{String}(n)$ is not any more $\mathfrak{so}(n)$, but a central extension of $\mathfrak{so}(n)$ by the abelian Lie 2-algebra $\mathbb{R} \rightarrow 0$, which is a Lie 2-algebra by itself¹.

To provide a way to characterize Lie 2-algebra structures on a 2-vector space, we categorify Weinstein's omni-Lie algebra $\mathfrak{gl}(V) \oplus V$ associated to a vector space V . The result is the so-called omni-Lie 2-algebra (Definition 3.3) $\mathfrak{gl}(\mathbb{V}) \oplus \mathbb{V}$ associated to a 2-vector space \mathbb{V} . We prove that there is a one-to-one correspondence between Dirac structures of the omni-Lie 2-algebra $\mathfrak{gl}(\mathbb{V}) \oplus \mathbb{V}$ and strict Lie 2-algebra structures on 2-sub-vector spaces of \mathbb{V} . We also introduce the notion of μ -twisted omni-Lie 2-algebra $\mathfrak{gl}(\mathbb{V}) \oplus_{\mu} \mathbb{V}$ twisted by an isomorphism μ from $\mathfrak{gl}(\mathbb{V})$ to itself. Dirac structures of the twisted omni-Lie 2-algebra $\mathfrak{gl}(\mathbb{V}) \oplus_{\mu} \mathbb{V}$ characterize those Lie 2-algebra structures on \mathbb{V} whose Jacobiators are determined in a specific way by the brackets. We further verify that an interesting class of Lie 2-algebras including string Lie 2-algebras is characterized by Dirac structures.

The paper is organized as following: In Section 2, we recall some necessary background knowledge. We construct the strict Lie 2-algebra $\mathfrak{gl}(\mathbb{V})$ for a 2-vector space \mathbb{V} , which plays the role of $\mathfrak{gl}(V)$ in the classical case for a vector space V . In Section 3, we introduce the notion of omni-Lie 2-algebra associated to a 2-vector space \mathbb{V} . An omni-Lie 2-algebra is the 2-vector space $\mathfrak{gl}(\mathbb{V}) \oplus \mathbb{V}$ together with some algebraic structures. We prove that Dirac structures of the omni-Lie 2-algebra $\mathfrak{gl}(\mathbb{V}) \oplus \mathbb{V}$ characterize strict Lie 2-algebra structures on 2-sub-vector spaces of \mathbb{V} (Theorem 3.9, 3.10). As an application of our theory, in Section 4, we introduce the notion of normalizer of a 2-sub-vector space of $\mathfrak{gl}(\mathbb{V}) \oplus \mathbb{V}$. We prove that the normalizer of a Dirac structure \mathbb{L} is a sub-Lie 2-algebra of $\mathfrak{gl}(\mathbb{V})$ and it can be considered as the derivation Lie 2-algebra (a la Schlessinger-Stasheff and Stevenson) of the Lie 2-algebra corresponding to \mathbb{L} . In Section 5, we introduce the notion of μ -twisted omni-Lie 2-algebras by an automorphism μ of $\mathfrak{gl}(\mathbb{V})$. We give the relation between Dirac structures of a μ -twisted omni-Lie 2-algebra $\mathfrak{gl}(\mathbb{V}) \oplus_{\mu} \mathbb{V}$ and Lie 2-algebra structures on the 2-vector space \mathbb{V} (Theorem 5.6). Finally, we give a description of Dirac structures corresponding to String Lie 2-algebras with a suitable choice of automorphism μ .

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2 Lie 2-algebras

Vector spaces can be categorified to 2-vector spaces. A good introduction for this subject is [1]. Let Vect be the category of vector spaces.

Definition 2.1. [1] A 2-vector space is a category in the category Vect .

Thus a 2-vector space C is a category with a vector space of objects C_0 and a vector space of morphisms C_1 , such that all the structure maps are linear. Let $s, t : C_1 \rightarrow C_0$ be the source and target maps respectively. Let \cdot_{\vee} be the composition of morphisms.

A 2-sub-vector space of C is a 2-vector space C' of which the set of morphisms C'_1 is a sub-vector space of C_1 , the set of objects C'_0 is a sub-vector space of C_0 , and all the structure maps are the restrictions of the corresponding structure maps of C .

It is well known that the 2-category of 2-vector spaces is equivalent to the 2-category of 2-term complexes of vector spaces. Roughly speaking, given a 2-vector space C , $\text{Ker}(s) \xrightarrow{t} C_0$ is a 2-term complex. Conversely, any 2-term complex of vector spaces $V_1 \xrightarrow{d} V_0$ gives rise to a 2-vector space of which the set of objects is V_0 , the set of morphisms is $V_0 \oplus V_1$, the source map s is given by

¹The concept of string Lie 2-algebra is later generalized to any such extension of a semisimple Lie algebra.

$s(v + m) = v$, and the target map t is given by $t(v + m) = v + dm$, where $v \in V_0$, $m \in V_1$. We denote the 2-vector space associated to the 2-term complex of vector spaces $V_1 \xrightarrow{d} V_0$ by \mathbb{V} :

$$\begin{array}{c} \mathbb{V}_1 := V_0 \oplus V_1 \\ \mathbb{V} = \quad \downarrow s \quad \downarrow t \\ \mathbb{V}_0 := V_0. \end{array} \quad (1)$$

Definition 2.2. A Lie 2-algebra is a 2-vector space C equipped with

- a skew-symmetric bilinear functor, the bracket, $[\cdot, \cdot] : C \times C \longrightarrow C$,
- a skew-symmetric trilinear natural isomorphism, the Jacobiator,

$$J_{x,y,z} : [[x, y], z] \longrightarrow [x, [y, z]] + [[x, z], y],$$

such that the following Jacobiator identity is satisfied,

$$\begin{aligned} & J_{[w,x],y,z}([J_{w,x,z}, y] + 1)(J_{w,[x,z],y} + J_{[w,z],x,y} + J_{w,x,[y,z]}) \\ &= [J_{w,x,y}, z](J_{[w,y],x,z} + J_{w,[x,y],z})([J_{w,y,z}, x] + 1)([w, J_{x,y,z}] + 1). \end{aligned}$$

A Lie 2-algebra is called **strict** if the Jacobiator is the identity isomorphism.

Remark 2.3. This notion is called a semistrict Lie 2-algebra in [1]. A Lie 2-algebra in their paper should be skew-symmetric also up to a natural transformation. However, since we will not use this other notion, we make this simplification, which also coincides with Henriques' definition [5] of Lie 2-algebras via L_∞ -algebras. The relation between Courant algebroids and L_∞ -algebras is studied in [13].

Definition 2.4. An L_∞ -algebra is a graded vector space $L = L_0 \oplus L_1 \oplus \dots$ equipped with a system $\{l_k \mid 1 \leq k < \infty\}$ of linear maps $l_k : \wedge^k L \longrightarrow L$ with degree² $\deg(l_k) = k - 2$, such that the following relation with Koszul sign “Ksgn” is satisfied for all $n \geq 0$:

$$\sum_{i+j=n+1} (-1)^{i(j-1)} \sum_{\sigma} \text{sgn}(\sigma) \text{Ksgn}(\sigma) l_j(l_i(x_{\sigma(1)}, \dots, x_{\sigma(i)}), x_{\sigma(i+1)}, \dots, x_{\sigma(n)}) = 0. \quad (2)$$

The summation in this equation is taken over all $(i, n-i)$ -unshuffles with $1 \leq i \leq n-1$.

In particular, if the k -ary brackets are zero for all $k > 2$, we recover the usual notion of **differential graded Lie algebras** (DGLA). If L is concentrated in degrees $< n$, L is called an n -term L_∞ -algebra.

It is well known that the notion of Lie 2-algebra is equivalent to that of 2-term L_∞ -algebra. In particular, strict Lie 2-algebras are the same as 2-term differential graded Lie algebras (DGLA), or equivalently, crossed modules of Lie algebras. Given a strict Lie 2-algebra C , the corresponding 2-term DGLA is given by $\ker(s) \xrightarrow{t} C_0$. Conversely, given a 2-term DGLA $V_1 \xrightarrow{d} V_0$, the underlying 2-vector space of the corresponding strict Lie 2-algebra is given by (1). Moreover, the skew-symmetric bracket is given by

$$[u + m, v + n] = [u, v] + [u, n] + [m, v] + [m, n]_d, \quad \forall u, v \in V_0, m, n \in V_1, \quad (3)$$

² the exterior powers are interpreted in the graded sense.

where the bracket $[\cdot, \cdot]_d$ on V_1 is defined by

$$[m, n]_d \triangleq [dm, n]. \quad (4)$$

Given a 2-vector space \mathbb{V} , we define $\text{End}_d^0(\mathbb{V})$ by

$$\text{End}_d^0(\mathbb{V}) \triangleq \{(A_0, A_1) \in \mathfrak{gl}(V_0) \oplus \mathfrak{gl}(V_1) | A_0 \circ d = d \circ A_1\},$$

and define $\text{End}^1(\mathbb{V}) \triangleq \text{End}(V_0, V_1)$. Then we have,

Lemma 2.5. $\text{End}_d^0(\mathbb{V})$ is the space of linear functors from \mathbb{V} to \mathbb{V} .

Proof. Let (f_1, f_0) be a linear functor. Then f_1 , written in the form of a matrix of linear morphisms $V_{0,1} \rightarrow V_{0,1}$, has the following form,

$$f_1 = \begin{pmatrix} A_0 & B \\ 0 & A_1 \end{pmatrix}.$$

Therefore, for $u \in V_1$ and $m \in V_0$, we have

$$s \circ f_1(u, m) = s(A_0 u + Bm, A_1 m) = A_0 u + Bm.$$

On the other hand, we have

$$f_0 \circ s(u, m) = f_0(u).$$

Since a linear functor commutes with the source map and (u, m) is arbitrary, we have

$$f_0 = A_0, \quad B = 0.$$

Furthermore, we have

$$t f_1(u, m) = t(A_0 u, A_1 m) = A_0 u + d \circ A_1 m,$$

and

$$f_0 \circ t(u, m) = f_0(u + dm) = A_0 u + A_0 \circ dm.$$

By the condition that a linear functor commutes with the target map, we have

$$A_0 \circ d = d \circ A_1.$$

So any linear functor is of the form $(\begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix}, A_0)$, where $A_0 \circ d = d \circ A_1$.

Furthermore, it is not hard to see that any linear map $(\begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix}, A_0)$, where $A_0 \circ d = d \circ A_1$, preserves the identity morphisms and the composition of morphisms. Thus $\text{End}_d^0(\mathbb{V})$ is the space of linear functors from \mathbb{V} to \mathbb{V} . ■

There is a differential $\delta : \text{End}^1(\mathbb{V}) \longrightarrow \text{End}_d^0(\mathbb{V})$ given by

$$\delta(\phi) \triangleq \phi \circ d + d \circ \phi, \quad \forall \phi \in \text{End}^1(\mathbb{V}),$$

and a bracket operation $[\cdot, \cdot]$ given by the graded commutator. More precisely, for any $A = (A_0, A_1), B = (B_0, B_1) \in \text{End}_d^0(\mathbb{V})$ and $\phi \in \text{End}^1(\mathbb{V})$, $[\cdot, \cdot]$ is given by

$$[A, B] = A \circ B - B \circ A = (A_0 \circ B_0 - B_0 \circ A_0, A_1 \circ B_1 - B_1 \circ A_1),$$

and

$$[A, \phi] = A \circ \phi - \phi \circ A = A_1 \circ \phi - \phi \circ A_0. \quad (5)$$

These two operations make $\text{End}^1(\mathbb{V}) \xrightarrow{\delta} \text{End}_d^0(\mathbb{V})$ into a 2-term DGLA (proved in [15]), which we denote by $\text{End}(\mathbb{V})$. This DGLA plays an important role in the theory of representations of higher Lie algebras in [7]. The corresponding (strict) Lie 2-algebra of this 2-term DGLA, denoted by $\mathfrak{gl}(\mathbb{V})$, is given by

$$\mathfrak{gl}(\mathbb{V}) = \begin{array}{c} \text{End}_d^0(\mathbb{V}) \oplus \text{End}^1(\mathbb{V}) \\ s \downarrow t \\ \text{End}_d^0(\mathbb{V}). \end{array} \quad (6)$$

For any $A \in \text{End}_d^0(\mathbb{V})$, $\phi \in \text{End}^1(\mathbb{V})$, the source and the target maps are given by

$$s(A + \phi) = A \quad \text{and} \quad t(A + \phi) = A + \delta\phi;$$

the skew-symmetric bilinear functor $[\cdot, \cdot] : \mathfrak{gl}(\mathbb{V}) \times \mathfrak{gl}(\mathbb{V}) \longrightarrow \mathfrak{gl}(\mathbb{V})$ is given by

$$[A + \phi, B + \psi] = [A, B] + [\phi, \psi]_\delta + [A, \psi] + [\phi, B],$$

where $[\phi, \psi]_\delta$ is given by

$$[\phi, \psi]_\delta = [\delta\phi, \psi] = \phi \circ d \circ \psi - \psi \circ d \circ \phi.$$

The Lie 2-algebra $\mathfrak{gl}(\mathbb{V})$ plays the same role of $\mathfrak{gl}(V)$ in the classical case of a vector space V . Another example with a similar flavor is X. Zhu's Lie 2-algebra $\mathfrak{gl}(\mathcal{C})$ for any abelian category \mathcal{C} [20].

The Lie 2-algebra $\mathfrak{gl}(\mathbb{V})$ acts on \mathbb{V} naturally:

$$(A + \phi)(u + m) = A(u + m) + \phi(u + dm), \quad \forall A \in \text{End}_d^0(\mathbb{V}), \phi \in \text{End}^1(\mathbb{V}). \quad (7)$$

It is not hard to see that this action is a bilinear functor from $\mathfrak{gl}(\mathbb{V}) \times \mathbb{V}$ to \mathbb{V} .

The action (7) is a generalization of the usual representation of $\mathfrak{gl}(V)$ on a vector space V . There is a natural Lie algebra structure on $\mathfrak{gl}(V) \oplus V$ which is the semidirect product of $\mathfrak{gl}(V)$ and V . Similarly, for a 2-vector space \mathbb{V} , there is also a similar semidirect product strict Lie 2-algebra structure on $\mathfrak{gl}(\mathbb{V}) \oplus \mathbb{V}$. This fact is proved in the case of 2-term L_∞ -algebras in [15]. However, in the next section, we introduce another bracket on $\mathfrak{gl}(\mathbb{V}) \oplus \mathbb{V}$ which does not make it into a Lie 2-algebra.

3 Dirac structures of omni-Lie 2-algebras

On the direct sum $\mathfrak{gl}(\mathbb{V}) \oplus \mathbb{V}$, we can define a \mathbb{V} -valued nondegenerate symmetric pairing $\langle \cdot, \cdot \rangle$. On the space of morphisms, it is given by

$$\langle A + \phi + u + m, B + \psi + v + n \rangle \triangleq \frac{1}{2} ((A + \phi)(v + n) + (B + \psi)(u + m)). \quad (8)$$

On the space of objects, it is given by

$$\langle A + u, B + v \rangle \triangleq \frac{1}{2} (Av + Bu). \quad (9)$$

Lemma 3.1. *The \mathbb{V} -valued nondegenerate symmetric pairing $\langle \cdot, \cdot \rangle$ defined by (8) and (9) is a bilinear functor from $(\mathfrak{gl}(\mathbb{V}) \oplus \mathbb{V}) \times (\mathfrak{gl}(\mathbb{V}) \oplus \mathbb{V})$ to \mathbb{V} .*

Proof. Obviously, the pairing $\langle \cdot, \cdot \rangle : (\mathfrak{gl}(\mathbb{V}) \oplus \mathbb{V}) \times (\mathfrak{gl}(\mathbb{V}) \oplus \mathbb{V}) \rightarrow \mathbb{V}$ is a bilinear map on the space of objects and on the space of morphisms. To see that it is a bilinear functor, we need to prove that

- (a) it preserves the source and target maps;
- (b) it preserves identity morphisms;
- (c) it preserves the composition of morphisms.

Item (b) is obvious. Now we give the proof of item (a).

By (8), we have

$$\begin{aligned} s \langle A + \phi + u + m, B + \psi + v + n \rangle &= s \left(\frac{1}{2} ((A + \phi)(v + n) + (B + \psi)(u + m)) \right) \\ &= \frac{1}{2} s((Av + An + \phi(v + dn) + Bu + Bm + \psi(u + dm))) \\ &= \frac{1}{2} (Av + Bu). \end{aligned}$$

By (9), we have

$$\begin{aligned} \langle s(A + \phi + u + m), s(B + \psi + v + n) \rangle &= \langle A + u, B + v \rangle \\ &= \frac{1}{2} (Av + Bu) \\ &= s \langle A + \phi + u + m, B + \psi + v + n \rangle. \end{aligned}$$

Thus the pairing $\langle \cdot, \cdot \rangle$ preserves the source map.

Similarly, considering the target map, we have

$$\begin{aligned} t \langle A + \phi + u + m, B + \psi + v + n \rangle &= t \left(\frac{1}{2} ((A + \phi)(v + n) + (B + \psi)(u + m)) \right) \\ &= \frac{1}{2} t((Av + An + \phi(v + dn) + Bu + Bm + \psi(u + dm))) \\ &= \frac{1}{2} (Av + Bu + d \circ An + d \circ \phi(v + dn) + d \circ Bm + d \circ \psi(u + dm)), \end{aligned}$$

and

$$\begin{aligned} \langle t(A + \phi + u + m), t(B + \psi + v + n) \rangle &= \langle A + \delta\phi + u + dm, B + \delta\psi + v + dn \rangle \\ &= \frac{1}{2} ((A + \delta\phi)(v + dn) + (B + \delta\psi)(u + dm)) \\ &= \frac{1}{2} (Av + A \circ dn + d \circ \phi(v + dn) + Bu + B \circ dm + d \circ \psi(u + dm)). \end{aligned}$$

Since $A, B \in \text{End}_d^0(\mathbb{V})$ satisfy

$$d \circ A = A \circ d, \quad d \circ B = B \circ d, \tag{10}$$

the pairing $\langle \cdot, \cdot \rangle$ preserves the target map.

At last, we prove that the pairing also preserves the composition of morphisms. For any $e_1 = A + \phi + u + m$, $e'_1 = A' + \phi' + u' + m'$, $e_2 = B + \psi + v + n$, $e'_2 = B' + \psi' + v' + n'$ satisfying

$$t(e_1) = s(e'_1), \quad t(e_2) = s(e'_2).$$

We have

$$A' = A + \delta\phi, \quad u' = u + dm, \quad (11)$$

$$B' = B + \delta\psi, \quad v' = v + dn, \quad (12)$$

$$e_1 \cdot_v e'_1 = A + \phi + \phi' + u + m + m',$$

and

$$e_2 \cdot_v e'_2 = B + \psi + \psi' + v + n + n'.$$

Thus we have

$$\begin{aligned} \langle e_1 \cdot_v e'_1, e_2 \cdot_v e'_2 \rangle &= \langle A + \phi + \phi' + u + m + m', B + \psi + \psi' + v + n + n' \rangle \\ &= \frac{1}{2}((A + \phi + \phi')(v + n + n') + (B + \psi + \psi')(u + m + m')). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \langle e_1, e_2 \rangle &= \langle A + \phi + u + m, B + \psi + v + n \rangle \\ &= \frac{1}{2}((A + \phi)(v + n) + (B + \psi)(u + m)), \end{aligned}$$

and

$$\begin{aligned} \langle e'_1, e'_2 \rangle &= \langle A' + \phi' + u' + m', B' + \psi' + v' + n' \rangle \\ &= \frac{1}{2}((A' + \phi')(v' + n') + (B' + \psi')(u' + m')). \end{aligned}$$

By straightforward computations, we have

$$\begin{aligned} t \langle e_1, e_2 \rangle &= \frac{1}{2}t((A + \phi)(v + n) + (B + \psi)(u + m)) \\ &= \frac{1}{2}(Av + d \circ An + d \circ \phi(v + dn) + Bu + d \circ Bm + d \circ \psi(u + dm)), \end{aligned}$$

and

$$\begin{aligned} s \langle e'_1, e'_2 \rangle &= \frac{1}{2}s((A' + \phi')(v' + n') + (B' + \psi')(u' + m')) \\ &= \frac{1}{2}(A'v' + B'u') \\ &= \frac{1}{2}((A + \delta\phi)(v + dn) + (B + \delta\phi)(u + dm)). \end{aligned}$$

By (10), since $\delta\phi = d \circ \phi$ on V_0 , we have

$$t \langle e_1, e_2 \rangle = s \langle e'_1, e'_2 \rangle.$$

It is not hard to see that

$$\begin{aligned} \langle e_1, e_2 \rangle \cdot_v \langle e'_1, e'_2 \rangle &= \frac{1}{2}((A + \phi)(v + n) + A'n' + \phi'(v' + n') \\ &\quad + (B + \psi)(u + m) + B'm' + \phi'(u' + m')). \end{aligned}$$

By (11), (12), and the definition of $\mathfrak{gl}(\mathbb{V})$ action (7), we have

$$A'n' = (A + \delta\phi)(n') = (A + \phi \circ d)(n') = (A + \phi)(n'),$$

and

$$\phi'(v' + n') = \phi'(v + dn + n') = \phi'(v + n + n').$$

After similar computations for $B'm' + \phi'(u' + m')$, we obtain that

$$\langle e_1, e_2 \rangle \cdot_v \langle e'_1, e'_2 \rangle = \langle e_1 \cdot_v e'_1, e_2 \cdot_v e'_2 \rangle,$$

which completes the proof. ■

Similar to Weinstein's definition of the bracket on omni-Lie algebras, we introduce a skew-symmetric bilinear bracket operation, denote by $[\cdot, \cdot]$, on $\mathfrak{gl}(\mathbb{V}) \oplus \mathbb{V}$. On the space of morphisms, it is given by

$$\begin{aligned} &[[A + \phi + u + m, B + \psi + v + n]] \\ &= [A + \phi, B + \psi] + \frac{1}{2}((A + \phi)(v + n) - (B + \psi)(u + m)). \end{aligned} \tag{13}$$

On the space of objects, it is given by

$$[[A + u, B + v]] = [A, B] + \frac{1}{2}(Av - Bu). \tag{14}$$

Lemma 3.2. *The bracket operation $[\cdot, \cdot]$ defined by (13) and (14) is a bilinear functor.*

Proof. It is straightforward to see that

$$\begin{aligned} &s[[A + \phi + u + m, B + \psi + v + n]] \\ &= s[A + \phi, B + \psi] + \frac{1}{2}s((A + \phi)(v + n) - (B + \psi)(u + m)) \\ &= [A, B] + \frac{1}{2}(Av - Bu). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} [[s(A + \phi + u + m), s(B + \psi + v + n)]] &= [[A + u, B + v]] \\ &= [A, B] + \frac{1}{2}(Av - Bu). \end{aligned}$$

Therefore, the bracket operation $[\cdot, \cdot]$ preserves the source map.

Now for the target map, we have

$$\begin{aligned} &t[[A + \phi + u + m, B + \psi + v + n]] \\ &= t[A + \phi, B + \psi] + \frac{1}{2}t((A + \phi)(v + n) - (B + \psi)(u + m)) \\ &= t[A + \phi, B + \psi] + \frac{1}{2}(Av + A \circ dn + d \circ \phi(v + dn) - Bu - d \circ Bm - d\psi(u + dm)), \end{aligned}$$

and

$$\begin{aligned} & \llbracket t(A + \phi + u + m), t(B + \psi + v + n) \rrbracket \\ &= \llbracket t(A + \phi) + u + dm, t(B + \psi) + v + dn \rrbracket \\ &= [t(A + \phi), t(B + \psi)] + \frac{1}{2}((A + \delta\phi)(v + dn) - (B + \delta\psi)(u + dm)). \end{aligned}$$

Since the bracket operation $[\cdot, \cdot]$ on $\mathfrak{gl}(\mathbb{V})$ is a bilinear functor, we have

$$[t(A + \phi), t(B + \psi)] = t[A + \phi, B + \psi].$$

By (10), we see that the bracket operation $\llbracket \cdot, \cdot \rrbracket$ also preserves the target map.

It is obvious that $\llbracket \cdot, \cdot \rrbracket$ preserves the identity morphism. Similar to the proof of Lemma 3.1, one can show that $\llbracket \cdot, \cdot \rrbracket$ also preserves the composition of morphisms. Thus the bracket operation $\llbracket \cdot, \cdot \rrbracket$ is a bilinear functor. ■

Since the nondegenerate symmetric pairing $\langle \cdot, \cdot \rangle$ and the bracket operation $\llbracket \cdot, \cdot \rrbracket$ are bilinear functors, they are totally determined by the values on the space of morphisms, i.e. they are determined by (8) and (13).

Definition 3.3. *The triple $(\mathfrak{gl}(\mathbb{V}) \oplus \mathbb{V}, \langle \cdot, \cdot \rangle, \llbracket \cdot, \cdot \rrbracket)$ is called the omni-Lie 2-algebra associated to the 2-vector space \mathbb{V} , where $\langle \cdot, \cdot \rangle$ is the symmetric bilinear functor given by (8) and $\llbracket \cdot, \cdot \rrbracket$ is the skew-symmetric bilinear functor given by (13). We simply denote the omni-Lie 2-algebra by $\mathfrak{gl}(\mathbb{V}) \oplus \mathbb{V}$.*

The factor of $\frac{1}{2}$ in $\llbracket \cdot, \cdot \rrbracket$ spoils the Jacobi identity. Computing the Jacobi identity of the bracket operation $\llbracket \cdot, \cdot \rrbracket$ on the space of objects, we have

$$\llbracket \llbracket A + u, B + v \rrbracket, C + w \rrbracket + c.p. = \frac{1}{4}([A, B]w + [B, C]u + [C, A]v).$$

Thus in general the omni-Lie 2-algebra $(\mathfrak{gl}(\mathbb{V}) \oplus \mathbb{V}, \langle \cdot, \cdot \rangle, \llbracket \cdot, \cdot \rrbracket)$ is not a Lie 2-algebra, since the Jacobiator identity is not satisfied.

We can also introduce another bracket operation $\{\cdot, \cdot\}$ on $\mathfrak{gl}(\mathbb{V}) \oplus \mathbb{V}$, which is not skew-symmetric, by setting

$$\{e_1, e_2\} \triangleq \llbracket e_1, e_2 \rrbracket + \langle e_1, e_2 \rangle. \quad (15)$$

Since $\llbracket \cdot, \cdot \rrbracket$ and $\langle \cdot, \cdot \rangle$ are all bilinear functors, $\{\cdot, \cdot\}$ is also a bilinear functor. Assume $e_1 = A + \phi + u + m$ and $e_2 = B + \psi + v + n$, we have

$$\{A + \phi + u + m, B + \psi + v + n\} = [A + \phi, B + \psi] + (A + \phi)(v + n). \quad (16)$$

By straightforward computations, we have

Proposition 3.4. *The bracket operation $\{\cdot, \cdot\}$ satisfies the Leibniz rule, i.e. for any $e_1, e_2, e_3 \in \mathfrak{gl}(\mathbb{V}) \oplus \mathbb{V}$, we have*

$$\{e_1, \{e_2, e_3\}\} = \{\{e_1, e_2\}, e_3\} + \{e_2, \{e_1, e_3\}\}.$$

Remark 3.5. *In [14], the first two authors introduce the notion of Leibniz 2-algebras, which is equivalent to 2-term Leibniz $_{\infty}$ -algebras [17]. In fact, Proposition 3.4 implies that $(\mathfrak{gl}(\mathbb{V}) \oplus \mathbb{V}, \{\cdot, \cdot\})$ is a strict Leibniz 2-algebras, please see [14] for more details.*

For a 2-sub-vector space $\mathbb{L} \subset \mathfrak{gl}(\mathbb{V}) \oplus \mathbb{V}$, define \mathbb{L}^\perp by

$$\mathbb{L}^\perp = \{e \in \mathfrak{gl}(\mathbb{V}) \oplus \mathbb{V} \mid \langle e, l \rangle = 0, \forall l \in \mathbb{L}\}. \quad (17)$$

Dirac structures of the omni-Lie 2-algebra $(\mathfrak{gl}(\mathbb{V}) \oplus \mathbb{V}, \langle \cdot, \cdot \rangle, [\![\cdot, \cdot]\!])$ are defined in the usual way.

Definition 3.6. A Dirac structure of the omni-Lie 2-algebra $(\mathfrak{gl}(\mathbb{V}) \oplus \mathbb{V}, \langle \cdot, \cdot \rangle, [\![\cdot, \cdot]\!])$ is a maximal isotropic 2-sub-vector space, i.e. $\mathbb{L} = \mathbb{L}^\perp$, which is closed under the bracket operation $[\![\cdot, \cdot]\!]$.

Proposition 3.7. Let D be a Dirac structure of the omni-Lie 2-algebra $\mathfrak{gl}(\mathbb{V}) \oplus \mathbb{V}$. Then $(D, [\![\cdot, \cdot]\!])$ is a strict Lie 2-algebra.

Proof. By (15), a maximal isotropic 2-sub-vector space is closed under $[\![\cdot, \cdot]\!]$ if and only if it is closed under $\{\cdot, \cdot\}$. By (15) and Proposition 3.4, $(D, [\![\cdot, \cdot]\!]) = (D, \{\cdot, \cdot\})$ is a strict Lie 2-algebra. ■

For any bilinear functor $F : \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{V}$, define $\text{ad}_F : \mathbb{V} \rightarrow \mathfrak{gl}(\mathbb{V})$ by

$$\text{ad}_F \xi(\eta) = F(\xi, \eta), \quad \forall \xi, \eta \in \mathbb{V}.$$

Lemma 3.8. For any bilinear functor $F : \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{V}$, the induced map $\text{ad}_F : \mathbb{V} \rightarrow \mathfrak{gl}(\mathbb{V})$ is a linear functor.

Proof. Since F is a bilinear functor, it is straightforward to see that

$$\text{ad}_F u \in \text{End}_d^0(\mathbb{V}), \quad \text{ad}_F m \in \text{End}^1(\mathbb{V}), \quad \forall u \in V_0, m \in V_1,$$

which implies that ad_F preserves the identity morphisms. Furthermore, we have

$$s(\text{ad}_F(u + m)) = \text{ad}_F u = \text{ad}_F(s(u + m)),$$

which implies that ad_F preserves the source map. For the target map, we have

$$\begin{aligned} t(\text{ad}_F(u + m)) &= \text{ad}_F u + \delta(\text{ad}_F m), \\ \text{ad}_F(t(u + m)) &= \text{ad}_F u + \text{ad}_F d m, \\ \delta(\text{ad}_F m)(v) &= d(\text{ad}_F m(v)) = dF(m, v), \quad \forall v \in V_0, \\ \delta(\text{ad}_F m)(n) &= \text{ad}_F m(d n) = F(m, d n), \quad \forall n \in V_1. \end{aligned}$$

According to Theorem 4.3.6 in [1] and Proposition 2.6 in [12], for any bilinear functor $F : \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{V}$, we have

$$dF(m, v) = F(dm, v), \quad F(m, dn) = F(dm, n).$$

Therefore, we have

$$\delta(\text{ad}_F m) = \text{ad}_F dm, \quad (18)$$

which implies that ad_F preserves the target map.

Finally, we prove that ad_F also preserves the composition of morphisms. For any morphisms $u + m$ and $v + n$ satisfying $t(u + m) = s(v + n)$, i.e. $v = u + dm$, we have

$$(u + m) \cdot_v (v + n) = u + m + n.$$

It is straightforward to see that

$$\text{ad}_F((u + m) \cdot_v (v + n)) = \text{ad}_F(u + m + n) = \text{ad}_F u + \text{ad}_F(m + n).$$

By (18), we have

$$tad_F(u + m) = \text{ad}_F u + \delta \text{ad}_F m = \text{ad}_F u + \text{ad}_F \text{dm} = \text{ad}_F(u + dm).$$

Obviously we have

$$s\text{ad}_F(v + n) = \text{ad}_F v = \text{ad}_F(u + dm).$$

Thus we have $s\text{ad}_F(v + n) = tad_F(u + m)$ and

$$\text{ad}_F(u + m) \cdot_v \text{ad}_F(v + n) = \text{ad}_F u + \text{ad}_F m + \text{ad}_F n = \text{ad}_F u + \text{ad}_F(m + n),$$

which yields that ad_F preserves the composition of morphisms. ■

We denote by $\mathfrak{G}_F \subset \mathfrak{gl}(\mathbb{V}) \oplus \mathbb{V}$ the graph of the operator ad_F .

Theorem 3.9. *Given a bilinear functor $F : \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{V}$, the graph \mathfrak{G}_F is a Dirac structure of the omni-Lie 2-algebra $\mathfrak{gl}(\mathbb{V}) \oplus \mathbb{V}$ if and only if F defines a strict Lie 2-algebra structure on the 2-vector space \mathbb{V} .*

Proof. For any $\xi, \eta \in \mathbb{V}$, we have

$$\langle \text{ad}_F \xi + \xi, \text{ad}_F \eta + \eta \rangle = \frac{1}{2} (\text{ad}_F \xi(\eta) + \text{ad}_F \eta(\xi)) = \frac{1}{2} (F(\xi, \eta) + F(\eta, \xi)).$$

Thus \mathfrak{G}_F is isotropic iff F is skew-symmetric. Take any $(A, \eta) \in \mathfrak{G}_F^\perp$, if F is skew-symmetric, then

$$0 = 2 \langle \text{ad}_F \xi + \xi, A + \eta \rangle = F(\xi, \eta) + A(\xi) = -\text{ad}_F \eta(\xi) + A(\xi).$$

This implies that $A = \text{ad}_F \eta$, thus $(A, \eta) \in \mathfrak{G}_F$. Hence \mathfrak{G}_F is actually maximal isotropic. By this argument, we see that \mathfrak{G}_F is maximal isotropic iff F is skew-symmetric.

Moreover, for any skew-symmetric bilinear functor F , we have

$$\begin{aligned} [\text{ad}_F \xi + \xi, \text{ad}_F \eta + \eta] &= [\text{ad}_F \xi, \text{ad}_F \eta] + \frac{1}{2} (\text{ad}_F \xi(\eta) - \text{ad}_F \eta(\xi)) \\ &= [\text{ad}_F \xi, \text{ad}_F \eta] + F(\xi, \eta). \end{aligned}$$

Thus \mathfrak{G}_F being a Dirac structure is equivalent to the fact that F is skew-symmetric and satisfies

$$[\text{ad}_F \xi, \text{ad}_F \eta] = \text{ad}_F F(\xi, \eta). \quad (19)$$

These two properties are also equivalent to the fact that F provides a strict Lie 2-algebra structure on \mathbb{V} : indeed, applying (19) to an element $\theta \in \mathbb{V}$ gives the Jacobi identity

$$F(\xi, F(\eta, \theta)) = F(F(\xi, \eta), \theta) + F(\eta, F(\xi, \theta)),$$

provided that F is skew-symmetric. ■

In [3], Chen and Liu introduce the notion of omni-Lie algebroid associated to a vector bundle E , which generalizes Weinstein's omni-Lie algebras associated a vector space V . They continue to study Dirac structures of omni-Lie algebroids in [4]. In that paper, they show that there is a one-to-one correspondence between Dirac structures (not necessarily coming from graphs) and projective Lie algebroids. This implies that, in particular, for the omni-Lie algebra $\mathfrak{gl}(V) \oplus V$, there is a one-to-one correspondence between Dirac structures (not necessarily coming from graphs) and Lie algebra structures on the sub-vector space of V .

For omni-Lie 2-algebras, we have

Theorem 3.10. *There is a one-to-one correspondence between Dirac structures of the omni-Lie 2-algebra $(\mathfrak{gl}(\mathbb{V}) \oplus \mathbb{V}, \langle \cdot, \cdot \rangle, [\cdot, \cdot])$ and strict Lie 2-algebra structures on 2-sub-vector spaces of \mathbb{V} .*

To prove this theorem, we need to adapt the theory of characteristic pairs developed in [8] for Courant algebroids to our setting. This theory has many applications in the theory of reduction of various geometric structures, see for example [10].

Given a maximal isotropic 2-sub-vector space \mathbb{L} of the omni-Lie 2-algebra $(\mathfrak{gl}(\mathbb{V}) \oplus \mathbb{V}, \langle \cdot, \cdot \rangle, [\cdot, \cdot])$, let $\mathbb{D} = \mathbb{L} \cap \mathfrak{gl}(\mathbb{V})$. Obviously, \mathbb{D} is a 2-sub-vector space. We define the 2-sub-vector space $\mathbb{D}^0 \subset \mathbb{V}$ to be the null space of \mathbb{D} :

$$\mathbb{D}^0 = \{\xi \in \mathbb{V} \mid X(\xi) = 0, \quad \forall X \in \mathbb{D}\}.$$

Similarly, for any 2-sub-vector space $\mathbb{W} \subset \mathbb{V}$, we define \mathbb{W}^0 by

$$\mathbb{W}^0 = \{D \in \mathfrak{gl}(\mathbb{V}) \mid D(\xi) = 0, \quad \forall \xi \in \mathbb{W}\}.$$

It is straightforward to see that

Lemma 3.11. *With the above notations, we have*

$$D \subset (D^0)^0, \quad (\mathbb{W}^0)^0 = \mathbb{W}.$$

If \mathbb{L} is maximal isotropic, then \mathbb{L} is of the form

$$\mathbb{L} = \mathbb{D} \oplus \mathfrak{G}_{\pi|_{\mathbb{D}^0}} = \{X + \pi(\xi) + \xi \mid X \in \mathbb{D}, \xi \in \mathbb{D}^0\},$$

for some linear functor $\pi : \mathbb{V} \rightarrow \mathfrak{gl}(\mathbb{V})$ satisfying

$$\pi(\xi)(\eta) = -\pi(\eta)(\xi). \tag{20}$$

Let us explain this: \mathbb{D} is the kernel of the projection $\text{pr}_{\mathbb{V}} : \mathfrak{gl}(\mathbb{V}) \oplus \mathbb{V} \rightarrow \mathbb{V}$ restricted to \mathbb{L} . Take any splitting $\mathbb{L} = \mathbb{D} \oplus \mathbb{D}'$. Denote by \mathbb{H} the image of $\text{pr}_{\mathbb{V}}|_{\mathbb{L}}$. Then $\text{pr}_{\mathbb{V}} : \mathbb{D}' \rightarrow \mathbb{H}$ is bijective, thus \mathbb{D}' is the graph of some linear functor, $\pi : \mathbb{V} \rightarrow \mathfrak{gl}(\mathbb{V})$, restricted to \mathbb{H} . Therefore, we have

$$\mathbb{L} = \mathbb{D} \oplus \mathfrak{G}_{\pi|_{\mathbb{H}}}.$$

For any $Y \in \mathfrak{gl}(\mathbb{V})$, since $\langle X + \pi(\xi) + \xi, Y \rangle = \frac{1}{2}Y(\xi)$, it is easy to see that Y is in \mathbb{H}^0 iff Y is in \mathbb{L}^\perp . Since \mathbb{L} is maximal isotropic, we have $\mathbb{L}^\perp = \mathbb{L}$. Thus $\mathbb{H}^0 = \mathbb{D}$, which implies that $\mathbb{H} = \mathbb{D}^0$. For any $\eta, \xi \in \mathbb{D}^0$, since \mathbb{L} is isotropic, we have

$$\langle \pi(\xi) + \xi, \pi(\eta) + \eta \rangle = \frac{1}{2}(\pi(\xi)(\eta) + \pi(\eta)(\xi)) = 0,$$

which implies that $\pi(\xi)(\eta) = -\pi(\eta)(\xi)$.

Clearly the function π depends on the choice of the splitting $\mathbb{L} = \mathbb{D} \oplus \mathbb{D}'$. Such a pair (\mathbb{D}, π) is called a **characteristic pair** of the maximal isotropic 2-sub-vector space \mathbb{L} . Two characteristic pairs (\mathbb{D}_1, π_1) and (\mathbb{D}_2, π_2) determine the same maximal isotropic 2-sub-vector space \mathbb{L} iff

$$\mathbb{D}_1 = \mathbb{D}_2, \quad \pi_1(\xi) - \pi_2(\xi) \in \mathbb{D}, \quad \forall \xi \in \mathbb{D}^0.$$

The conditions under which $\mathbb{L} = \mathbb{D} \oplus \mathfrak{G}_{\pi|_{\mathbb{D}^0}}$ is a Dirac structure is given by the following proposition.

Proposition 3.12. Let (\mathbb{D}, π) be a characteristic pair of a maximal isotropic 2-sub-vector space \mathbb{L} of $\mathfrak{gl}(\mathbb{V}) \oplus \mathbb{V}$. Then $\mathbb{L} = \mathbb{D} \oplus \mathfrak{G}_{\pi|_{\mathbb{D}^0}}$ is a Dirac structure if and only if for any $\xi, \eta \in \mathbb{D}^0$, the following conditions are satisfied:

- (1) \mathbb{D} is a sub-Lie 2-algebra of $\mathfrak{gl}(\mathbb{V})$;
- (2) $\pi(\pi(\xi)(\eta)) - [\pi(\xi), \pi(\eta)] \in \mathbb{D}$;
- (3) $\pi(\xi)(\eta) \in \mathbb{D}^0$.

Proof. Suppose that \mathbb{L} is a Dirac structure. For any $X, Y \in \mathbb{D}$, we have $[\![X, Y]\!] = [X, Y]$. Thus \mathbb{L} being a Dirac structure implies that $[X, Y] \in \mathbb{L} \cap \mathfrak{gl}(\mathbb{V}) = \mathbb{D}$, so \mathbb{D} is a sub-Lie 2-algebra of $\mathfrak{gl}(\mathbb{V})$. For any $X, Y \in \mathbb{D}$, $\eta \in \mathbb{D}^0$, we have

$$[\![X, Y + \pi(\eta) + \eta]\!] = [X, Y] + [X, \pi(\eta)] + \frac{1}{2}X(\eta) = [X, Y] + [X, \pi(\eta)] \in \mathbb{L},$$

which implies that $[X, \pi(\eta)] \in \mathbb{L} \cap \mathfrak{gl}(\mathbb{V}) = \mathbb{D}$. Similarly, for any $\xi, \eta \in \mathbb{D}^0$, we have

$$[\![\pi(\xi) + \xi, \pi(\eta) + \eta]\!] = [\pi(\xi), \pi(\eta)] + \pi(\xi)(\eta) \in \mathbb{L},$$

which implies that $\pi(\xi)(\eta) \in \mathbb{D}^0$ and $[\pi(\xi), \pi(\eta)] - \pi(\pi(\xi)(\eta)) \in \mathbb{D}$.

Conversely, for any $X, Y \in \mathbb{D}$, $\xi, \eta \in \mathbb{D}^0$, we have

$$[\![X + \pi(\xi) + \xi, Y + \pi(\eta) + \eta]\!] = [X, Y] + [X, \pi(\eta)] + [\pi(\xi), Y] + [\pi(\xi), \pi(\eta)] + \pi(\xi)(\eta).$$

Then for any $\theta \in \mathbb{D}^0$, $[\pi(\xi), Y](\theta) = \pi(\xi)(Y(\theta)) - Y(\pi(\xi)(\theta)) = 0$ by Condition (3). Thus $[\pi(\xi), Y] \in \mathbb{D}$. Then it is straightforward to see that \mathbb{L} is a Dirac structure if Conditions (1), (2), (3) are satisfied. This concludes the proof. ■

The proof of Theorem 3.10: For any Dirac structure $\mathbb{L} = \mathbb{D} \oplus \mathfrak{G}_{\pi|_{\mathbb{D}^0}}$, define a bilinear functor, $[\cdot, \cdot]_{\mathbb{D}^0} : \mathbb{D}^0 \times \mathbb{D}^0 \longrightarrow \mathbb{D}^0$ on \mathbb{D}^0 by

$$[\xi, \eta]_{\mathbb{D}^0} \triangleq \pi(\xi)(\eta), \quad \forall \xi, \eta \in \mathbb{D}^0.$$

By Condition (3) in Proposition 3.12, it is well defined, and by (20) it is skew-symmetric. By Condition (2), we have for all $\xi, \eta, \theta \in \mathbb{D}^0$,

$$\begin{aligned} [[\xi, \eta]_{\mathbb{D}^0}, \theta]_{\mathbb{D}^0} &= \pi([\xi, \eta]_{\mathbb{D}^0})(\theta) = \pi((\pi(\xi)(\eta))(\theta)) = [\pi(\xi), \pi(\eta)](\theta) \\ &= \pi(\xi)(\pi(\eta)(\theta)) - \pi(\eta)(\pi(\xi)(\theta)) \\ &= [\xi, [\eta, \theta]_{\mathbb{D}^0}]_{\mathbb{D}^0} - [\eta, [\xi, \theta]_{\mathbb{D}^0}]_{\mathbb{D}^0}, \end{aligned}$$

which implies that $(\mathbb{D}^0, [\cdot, \cdot]_{\mathbb{D}^0})$ is a strict Lie 2-algebra. Thus any Dirac structure gives rise to a strict Lie 2-algebra structure on a 2-sub-vector space of \mathbb{V} .

Conversely, for any 2-sub-vector space \mathbb{W} of \mathbb{V} , assume that $\mathbb{V} = \mathbb{W} \oplus \mathbb{W}'$. Define \mathbb{D} by

$$\mathbb{D} = \mathbb{W}^0 \triangleq \{X \in \mathfrak{gl}(\mathbb{V}) \mid X(\xi) = 0, \quad \forall \xi \in \mathbb{W}\}.$$

Then by Lemma 3.11, we have $\mathbb{D}^0 = \mathbb{W}$. Obviously, for any $D, D' \in \mathbb{D}$ and $\xi \in \mathbb{W}$, we have

$$[D, D'](\xi) = 0,$$

which implies that \mathbb{D} is a sub-Lie 2-algebra of $\mathfrak{gl}(\mathbb{V})$. By the inclusion $\mathbb{W} \rightarrow \mathbb{V}$, we have a natural embedding $\mathfrak{gl}(\mathbb{W}) \subset \mathfrak{gl}(\mathbb{V})$ as a sub-Lie 2-algebra. For any Lie 2-algebra structure $[\cdot, \cdot]_{\mathbb{W}}$ on \mathbb{W} , we

have a linear functor $\text{ad} : \mathbb{W} \longrightarrow \mathfrak{gl}(\mathbb{W})$ which is given by $\text{ad}_\xi(\eta) = [\xi, \eta]_{\mathbb{W}}$. Define $\mathcal{F} : \mathbb{V} \longrightarrow \mathfrak{gl}(\mathbb{V})$, as an extension of ad , by setting

$$\mathcal{F}(\xi + \xi') \triangleq \text{ad}_\xi, \quad \forall \xi \in \mathbb{W}, \xi' \in \mathbb{W}'.$$

Now we consider the 2-sub-vector space $\mathbb{L} \triangleq \mathbb{D} \oplus \mathfrak{G}_{\mathcal{F}|_{\mathbb{W}}}$, which is the direct sum of \mathbb{D} and the graph of $\mathcal{F}|_{\mathbb{W}}$. Since $[\cdot, \cdot]_{\mathbb{W}}$ is skew-symmetric, \mathbb{L} is isotropic. Take $A + \eta \in \mathbb{L}^\perp$. First by $\langle X, A + \eta \rangle = 0$ for all $X \in \mathbb{D}$, we have $\eta \in \mathbb{D}^0$. Thus for all $X \in \mathbb{D}, \xi \in \mathbb{D}^0$, we have

$$\langle X + \mathcal{F}(\xi) + \xi, A + \eta \rangle = \mathcal{F}(\xi)(\eta) + A\xi = -\mathcal{F}(\eta)(\xi) + A\xi = 0,$$

which implies that $A = \mathcal{F}(\eta) + Y$, for some $Y \in \mathbb{D}$. Thus $A + \eta \in \mathbb{D} + \mathfrak{G}_{\mathcal{F}|_{\mathbb{W}}}$. By the fact that $[\cdot, \cdot]_{\mathbb{W}}$ satisfies the Jacobi identity, we obtain

$$[\text{ad}_\xi, \text{ad}_\eta] = \text{ad}_{[\xi, \eta]_{\mathbb{W}}} = \text{ad}_{\text{ad}_\xi \eta}, \quad \forall \xi, \eta \in \mathbb{W}.$$

Furthermore, for any $D \in \mathbb{D}$ and $\xi, \eta \in \mathbb{W}$, it is obvious that

$$[D, \text{ad}_\xi](\eta) = D([\xi, \eta]_{\mathbb{W}}) - [\xi, D(\eta)] = 0,$$

which yields that $[D, \text{ad}_\xi] \in \mathbb{D}$. Therefore, we have

$$\begin{aligned} [[D + \text{ad}_\xi + \xi, D' + \text{ad}_\eta + \eta]] &= [D, D'] + [D, \text{ad}_\eta] + [\text{ad}_\xi, D'] + [\text{ad}_\xi, \text{ad}_\eta] + \frac{1}{2}(\text{ad}_\xi(\eta) - \text{ad}_\eta(\xi)) \\ &= [D, D'] + [D, \text{ad}_\eta] + [\text{ad}_\xi, D'] + \text{ad}_{[\xi, \eta]_{\mathbb{W}}} + [\xi, \eta]_{\mathbb{W}} \\ &\in \mathbb{D} \oplus \mathfrak{G}_{\mathcal{F}|_{\mathbb{W}}}, \end{aligned}$$

which yields that \mathbb{L} is closed under the bracket operation (13). Thus \mathbb{L} is a Dirac structure. Obviously a different extension \mathcal{F}' of ad gives rise to the same Dirac structure. ■

4 Normalizer of a Dirac structure

In this section, we introduce the notion of the normalizer of a 2-sub-vector space of the omni-Lie 2-algebra $\mathfrak{gl}(\mathbb{V}) \oplus \mathbb{V}$. In the classical case, the normalizer of the graph of the adjoint operator of a Lie algebra \mathfrak{g} is the derivation Lie algebra $\text{Der}(\mathfrak{g})$. Here we will prove a similar result for strict Lie 2-algebras.

Definition 4.1. *The normalizer of a 2-sub-vector space $\mathbb{K} \subset \mathfrak{gl}(\mathbb{V}) \oplus \mathbb{V}$ is composed of all the elements $N \in \mathfrak{gl}(\mathbb{V})$ such that*

$$\{N, \mathbb{K}\} \subset \mathbb{K},$$

with $\{\cdot, \cdot\}$ defined in (15).

Denote by $N_{\mathbb{K}}$ the normalizer of \mathbb{K} . Especially we care about the normalizer of a Dirac structure \mathbb{L} .

Proposition 4.2. *Let $\mathbb{L} = \mathbb{D} \oplus \mathfrak{G}_{\pi|_{\mathbb{D}^0}}$ be a Dirac structure of the omni-Lie 2-algebra $\mathfrak{gl}(\mathbb{V}) \oplus \mathbb{V}$ with characteristic pair (\mathbb{D}, π) . Then by Theorem 3.10, \mathbb{D}^0 is a strict Lie 2-algebra with Lie bracket $[\xi, \eta]_{\mathbb{D}^0} = \pi(\xi)(\eta)$. Now we claim that $N \in N_{\mathbb{L}}$ if and only if*

1. for all $X \in \mathbb{D}$, we have $[N, X] \in \mathbb{D}$,

2. for all $\xi, \eta \in \mathbb{D}^0 \subset \mathbb{V}$, we have $N(\xi) \in \mathbb{D}^0$ and $N([\xi, \eta]_{\mathbb{D}^0}) = [\xi, N(\eta)]_{\mathbb{D}^0} + [N(\xi), \eta]_{\mathbb{D}^0}$.

Proof. By definition, $N \in N_{\mathbb{L}}$ if and only if for all $X \in \mathbb{D}$ and $\xi \in \mathbb{D}^0$, we have

$$\{N, X + \pi(\xi) + \xi\} = [N, X] + [N, \pi(\xi)] + N(\xi) \in \mathbb{L},$$

which is equivalent to the fact that $[N, X] \in \mathbb{D}$ for all $X \in \mathbb{D}$ (by taking $\xi = 0$), and $N(\xi) \in \mathbb{D}^0$ and $\pi(N(\xi)) - [N, \pi(\xi)] \in \mathbb{D}$ for all $\xi \in \mathbb{D}^0$. The fact that $\pi(N(\xi)) - [N, \pi(\xi)] \in \mathbb{D}$ for all $\xi \in \mathbb{D}^0$, is equivalent to the fact that, for all $\eta, \xi \in \mathbb{D}^0$,

$$0 = \pi(N(\xi))(\eta) - [N, \pi(\xi)](\eta) = [N(\xi), \eta]_{\mathbb{D}^0} - N([\xi, \eta]_{\mathbb{D}^0}) + [\xi, N(\eta)]_{\mathbb{D}^0}.$$

This finishes the proof. ■

It is subtle to define sub-Lie 2-algebras of a Lie 2-algebra. At first sight, we might define a sub-Lie 2-algebra of a Lie 2-algebra to be a 2-sub-vector space which is closed under the bracket operation (Baez et al. use this definition in [2]). Then one can also propose that a sub-Lie 2-algebra L' of a Lie 2-algebra L is an injective morphism $\mu : L' \rightarrow L$ (see Definition 5.1). These two definitions are not the same. The second definition gives the first definition iff μ is strict. We clarify here that we use the first definition in this paper, even though in [15] we use the second more general definition.

Proposition 4.3. *The normalizer $N_{\mathbb{L}}$ of a Dirac structure \mathbb{L} of an omni-Lie 2-algebra $\mathfrak{gl}(\mathbb{V}) \oplus \mathbb{V}$ associated to a 2-vector space \mathbb{V} is a sub-Lie 2-algebra of $\mathfrak{gl}(\mathbb{V})$.*

Proof. First of all we show that $N_{\mathbb{L}}$ is a sub-2-vector space of $\mathfrak{gl}(\mathbb{V})$. For this, we only need to verify that $\delta(N_{\mathbb{L}} \cap \text{End}^1(\mathbb{V})) \subset N_{\mathbb{L}}$.

Take $\phi \in N_{\mathbb{L}} \cap \text{End}^1(\mathbb{V})$. By Proposition 4.2, and using the same notation therein, for all $X \in \mathbb{D}$, we have

$$[\delta(\phi), X] = \delta([\phi, X]) - [\phi, \delta(X)] \in \mathbb{D},$$

since $(\text{End}^1(\mathbb{V}) \xrightarrow{\delta} \text{End}_d^0(\mathbb{V}), [\cdot, \cdot])$ is a DGLA and \mathbb{D} is a 2-sub-vector space of $\mathfrak{gl}(\mathbb{V}) \oplus \mathbb{V}$. Then for all $X \in \mathbb{D}$, $\xi \in \mathbb{D}^0$, we have

$$X(\delta(\phi)(\xi)) = [X, \delta(\phi)](\xi) + \delta(\phi)(X(\xi)) = [X, \delta(\phi)](\xi) = 0.$$

Thus $\delta(\phi)(\xi) \in \mathbb{D}^0$. Now for all $u, v \in \mathbb{D}^0 \cap V_0$, $n \in \mathbb{D}^0 \cap V_1$, we have

$$\begin{aligned} \delta(\phi)([u, v]_{\mathbb{D}^0}) &= d \circ \phi([u, v]_{\mathbb{D}^0}) = d([\phi(u), v]_{\mathbb{D}^0}) + d([u, \phi(v)]_{\mathbb{D}^0}) \\ &= [d(\phi(u)), v]_{\mathbb{D}^0} + [u, d(\phi(v))]_{\mathbb{D}^0} \\ &= [\delta(\phi)(u), v]_{\mathbb{D}^0} + [u, \delta(\phi)(v)]_{\mathbb{D}^0}, \end{aligned}$$

and

$$\begin{aligned} \delta(\phi)([u, n]_{\mathbb{D}^0}) &= \phi \circ d([u, n]_{\mathbb{D}^0}) = \phi[d(u, n)]_{\mathbb{D}^0} \\ &= [\phi(d(u)), n]_{\mathbb{D}^0} + [d(u, \phi(n))]_{\mathbb{D}^0} \\ &= [\delta(\phi)(u), n]_{\mathbb{D}^0} + [u, \delta(\phi)(n)]_{\mathbb{D}^0}. \end{aligned}$$

Thus by Proposition 4.2, $\delta(\phi) \in N_{\mathbb{L}}$.

To see further that $N_{\mathbb{L}}$ is a sub-Lie 2-algebra, we only need to prove that $N_{\mathbb{L}}$ is closed under the induced bracket operation. For all $N, N' \in N_{\mathbb{L}}, l \in \mathbb{L}$, we have

$$\{\{N, N'\}, l\} = \{[N, N'], l\}.$$

On the other hand, by Proposition 3.4, we have

$$\{\{N, N'\}, l\} = \{N, \{N', l\}\} - \{N', \{N, l\}\} \in \mathbb{L}.$$

Therefore, $[N, N'] \in N_{\mathbb{L}}$. ■

As a corollary of the above two propositions, we have,

Corollary 4.4. *Given a strict Lie 2-algebra \mathbb{V} , the normalizer $N_{\mathfrak{G}_{\text{ad}}}$ of the Dirac structure \mathfrak{G}_{ad} is a sub-Lie 2-algebra of $\mathfrak{gl}(\mathbb{V})$, and $D \in N_{\mathfrak{G}_{\text{ad}}}$ if and only if for any $\xi, \eta \in \mathbb{V}$, we have*

$$D[\xi, \eta] = [D\xi, \eta] + [\xi, D\eta]. \quad (21)$$

Remark 4.5. *In some unpublished works³, Stevenson and Schlessinger-Stasheff study the notion of derivations of Lie n -algebras. Here by studying the normalizer of the graph of adjoint operator, we recover their notion of derivation: the Lie 2-algebra $N_{\mathfrak{G}_{\text{ad}}}$ is exactly the Lie 2-algebra of derivations $\text{Der}(\mathbb{V})$ for a Lie 2-algebra \mathbb{V} .*

5 Twisted omni-Lie 2-algebras

In this section we introduce the notion of twisted omni-Lie 2-algebra $\mathfrak{gl}(\mathbb{V}) \oplus_{\mu} \mathbb{V}$, where $\mu : \mathfrak{gl}(\mathbb{V}) \rightarrow \mathfrak{gl}(\mathbb{V})$ is an isomorphism of Lie 2-algebras. We will show that Dirac structures of the twisted omni-Lie 2-algebra $\mathfrak{gl}(\mathbb{V}) \oplus_{\mu} \mathbb{V}$ characterize some special Lie 2-algebra structures on \mathbb{V} .

Definition 5.1. [1] *Given Lie 2-algebras C and C' , a Lie 2-algebra morphism $\mu : C \rightarrow C'$ consists of:*

- a linear functor (μ_0, μ_1) from the underlying 2-vector space of C to that of C' , and
- a skew-symmetric bilinear natural transformation

$$\mu_2(u, v) : \mu_0[u, v] \rightarrow [\mu_0(u), \mu_0(v)]$$

such that the following diagram commutes:

$$\begin{array}{ccc}
\mu_0[[u, v], w] & \xrightarrow{\mu_0 J_{u, v, w}} & \mu_0[u, [v, w]] + \mu_0[v, [w, u]] \\
\downarrow \mu_2([u, v], w) & & \downarrow \mu_2(u, [v, w]) + \mu_2(v, [w, u]) \\
[\mu_0[u, v], \mu_0(w)] & & [\mu_0(u), \mu_0[v, w]] + [\mu_0(v), \mu_0[w, u]] \\
\downarrow [\mu_2(u, v), id_{\mu_0(w)}] & & \downarrow [id_{\mu_0(u)}, \mu_2(v, w)] + [id_{\mu_0(v)}, \mu_2(w, u)] \\
[[\mu_0(u), \mu_0(v)], \mu_0(w)] & \xrightarrow{J} & [\mu_0(u), [\mu_0(v), \mu_0(w)]] + [\mu_0(v), [\mu_0(w), \mu_0(u)]].
\end{array} \tag{22}$$

We call μ strict if $\mu_2 = 0$. We call μ an isomorphism if it induces an isomorphism of the underlying 2-vector spaces.

³Private conversation with Stevenson.

Now we take an isomorphism μ from the Lie 2-algebra $\mathfrak{gl}(\mathbb{V})$ to itself. We define the following μ -twisted bracket $\llbracket \cdot, \cdot \rrbracket_\mu$ on $\mathfrak{gl}(\mathbb{V}) \oplus \mathbb{V}$:

$$\begin{aligned} & \llbracket A + \phi + u + m, B + \psi + v + n \rrbracket_\mu \\ &= [A + \phi, B + \psi] + \frac{1}{2}(\mu_1(A + \phi)(v + n) - \mu_1(B + \psi)(u + m)). \end{aligned} \quad (23)$$

The nondegenerate \mathbb{V} -valued pairing can also be twisted by μ ,

$$\langle A + \phi + u + m, B + \psi + v + n \rangle_\mu = \frac{1}{2}(\mu_1(A + \phi)(v + n) + \mu_1(B + \psi)(u + m)). \quad (24)$$

Definition 5.2. *The triple $(\mathfrak{gl}(\mathbb{V}) \oplus \mathbb{V}, \llbracket \cdot, \cdot \rrbracket_\mu, \langle \cdot, \cdot \rangle_\mu)$ is called a μ -twisted omni-Lie 2-algebra. We simply denote it by $\mathfrak{gl}(\mathbb{V}) \oplus_\mu \mathbb{V}$.*

We also introduce the bracket $\{\cdot, \cdot\}_\mu$ by setting

$$\{\cdot, \cdot\}_\mu = \llbracket \cdot, \cdot \rrbracket_\mu + \langle \cdot, \cdot \rangle_\mu. \quad (25)$$

It is not hard to see that $\llbracket \cdot, \cdot \rrbracket_\mu$, $\langle \cdot, \cdot \rangle_\mu$ and $\{\cdot, \cdot\}_\mu$ are all bilinear functors since these operations without μ -twist are bilinear functors and μ is an isomorphism of the strict Lie 2-algebra $\mathfrak{gl}(\mathbb{V})$.

Proposition 5.3. *We have a natural transformation J between the functors $\{\{\cdot, \cdot\}_\mu, \cdot\}_\mu$ and $\{\cdot, \{\cdot, \cdot\}_\mu\}_\mu - \{\cdot, \{\cdot, \cdot\}_\mu\}_\mu$ defined as follows: For any objects $e_1 = A + u$, $e_2 = B + v$, $e_3 = C + w$ in $\mathfrak{gl}(\mathbb{V}) \oplus \mathbb{V}$,*

$$J_{e_1, e_2, e_3} : \{\{e_1, e_2\}_\mu, e_3\}_\mu \longrightarrow \{e_1, \{e_2, e_3\}_\mu\}_\mu - \{e_2, \{e_1, e_3\}_\mu\}_\mu,$$

is given by

$$J_{e_1, e_2, e_3} \triangleq [[A, B], C] + \mu_2(A, B)(w). \quad (26)$$

Proof. By straightforward computations, we have

$$\{\{e_1, e_2\}_\mu, e_3\}_\mu = [[A, B], C] + \mu_0[A, B](w),$$

and

$$\begin{aligned} & \{e_1, \{e_2, e_3\}_\mu\}_\mu - \{e_2, \{e_1, e_3\}_\mu\}_\mu \\ &= [A, [B, C]] + \mu_0(A)\mu_0(B)(w) - [B, [A, C]] - \mu_0(B)\mu_0(A)(w) \\ &= [A, [B, C]] - [B, [A, C]] + [\mu_0(A), \mu_0(B)](w). \end{aligned}$$

Since $[[A, B], C] = [A, [B, C]] - [B, [A, C]]$ and

$$\mu_2(A, B) : \mu_0[A, B] \longrightarrow [\mu_0(A), \mu_0(B)]$$

is a natural transformation, we conclude that J is also a natural transformation. ■

Definition 5.4. *A Dirac structure of the omni-Lie 2-algebra $\mathfrak{gl}(\mathbb{V}) \oplus_\mu \mathbb{V}$ is a maximal isotropic 2-sub-vector space (w.r.t. $\langle \cdot, \cdot \rangle_\mu$) closed under the bracket operation $\llbracket \cdot, \cdot \rrbracket_\mu$.*

Similar to the proof of Proposition 3.7, the following conclusion follows directly from Proposition 5.3.

Proposition 5.5. *Let \mathbb{L} be a Dirac structure of the omni-Lie 2-algebra $\mathfrak{gl}(\mathbb{V}) \oplus_{\mu} \mathbb{V}$. Then with J given in Proposition 5.3, $(\mathbb{L}, [\cdot, \cdot]_{\mu}|_{\mathbb{L}}, J)$ is a Lie 2-algebra.*

Given a linear functor \mathcal{F} from \mathbb{V} to $\mathfrak{gl}(\mathbb{V})$, we define a bilinear functor, $[\cdot, \cdot]_{\mu, \mathcal{F}} : \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{V}$ by

$$[\xi, \eta]_{\mu, \mathcal{F}} \triangleq \mu_1(\mathcal{F}(\xi))(\eta), \quad (27)$$

and a multilinear function $J : V_0 \times V_0 \times V_0 \rightarrow V_1$ by

$$J_{u, v, w} \triangleq \mu_2(\mathcal{F}(u), \mathcal{F}(v))(w), \quad \forall u, v, w \in V_0. \quad (28)$$

Theorem 5.6. *With the notation given above, the graph of a linear functor $\mathcal{F} : \mathbb{V} \rightarrow \mathfrak{gl}(\mathbb{V})$ is a Dirac structure of the twisted omni-Lie 2-algebra $\mathfrak{gl}(\mathbb{V}) \oplus_{\mu} \mathbb{V}$ if and only if $(\mathbb{V}, [\cdot, \cdot]_{\mu, \mathcal{F}}, J)$ is a Lie 2-algebra.*

Proof. For any $\xi, \eta \in \mathbb{V}$, we have

$$\langle \mathcal{F}(\xi) + \xi, \mathcal{F}(\eta) + \eta \rangle_{\mu} = \frac{1}{2}(\mu_1(\mathcal{F}(\xi))(\eta) + \mu_1(\mathcal{F}(\eta))(\xi)),$$

and

$$[\mathcal{F}(\xi) + \xi, \mathcal{F}(\eta) + \eta]_{\mu} = [\mathcal{F}(\xi), \mathcal{F}(\eta)] + \frac{1}{2}(\mu_1(\mathcal{F}(\xi))(\eta) - \mu_1(\mathcal{F}(\eta))(\xi)).$$

Therefore, similar to the proof of Theorem 3.9, the graph of \mathcal{F} is a Dirac structure iff

$$\mu_1(\mathcal{F}(\xi))(\eta) = -\mu_1(\mathcal{F}(\eta))(\xi),$$

and

$$[\mathcal{F}(\xi), \mathcal{F}(\eta)] = \mathcal{F}(\mu_1(\mathcal{F}(\xi))(\eta)) \equiv \mathcal{F}([\xi, \eta]_{\mu, \mathcal{F}}).$$

Now suppose that the graph of \mathcal{F} is a Dirac structure. For any $u, v, w \in V_0$, we have

$$[[u, v]_{\mu, \mathcal{F}}, w]_{\mu, \mathcal{F}} = \mu_1(\mathcal{F}([u, v]_{\mu, \mathcal{F}}))(w) = \mu_1([\mathcal{F}(u), \mathcal{F}(v)])(w).$$

On the other hand, we have

$$\begin{aligned} [u, [v, w]_{\mu, \mathcal{F}}]_{\mu, \mathcal{F}} - [v, [u, w]_{\mu, \mathcal{F}}]_{\mu, \mathcal{F}} &= \mu_1(\mathcal{F}(u))\mu_1(\mathcal{F}(v))(w) - \mu_1(\mathcal{F}(v))\mu_1(\mathcal{F}(u))(w) \\ &= [\mu_1(\mathcal{F}(u)), \mu_1(\mathcal{F}(v))](w). \end{aligned}$$

Since

$$\mu_2(\mathcal{F}(u), \mathcal{F}(v)) : \mu_1([\mathcal{F}(u), \mathcal{F}(v)]) \longrightarrow [\mu_1(\mathcal{F}(u)), \mu_1(\mathcal{F}(v))]$$

is a natural transformation, we obtain the skew-symmetric trilinear isomorphism

$$J_{u, v, w} : [[u, v]_{\mu, \mathcal{F}}, w]_{\mu, \mathcal{F}} \longrightarrow [u, [v, w]_{\mu, \mathcal{F}}]_{\mu, \mathcal{F}} - [v, [u, w]_{\mu, \mathcal{F}}]_{\mu, \mathcal{F}},$$

which is given by

$$J_{u, v, w} = \mu_2(\mathcal{F}(u), \mathcal{F}(v))(w). \quad (29)$$

Then the Jacobiator identity in Definition 2.2 holds because (μ_0, μ_1, μ_2) is a morphism from $\mathfrak{gl}(\mathbb{V})$ to itself. Thus $(\mathbb{V}, [\cdot, \cdot]_{\mu}, \mathcal{F}, J)$ is a Lie 2-algebra.

Conversely, if $(\mathbb{V}, [\cdot, \cdot]_\mu, \mathcal{F}, J)$ is a Lie 2-algebra, then

$$\mu_1(\mathcal{F}(\xi))(\eta) + \mu_1(\mathcal{F}(\eta))(\xi) = [\xi, \eta]_{\mu, \mathcal{F}} + [\eta, \xi]_{\mu, \mathcal{F}} = 0.$$

Thus we only need to show that $[\mathcal{F}(\xi), \mathcal{F}(\eta)] = \mathcal{F}([\xi, \eta]_{\mu, \mathcal{F}})$. On one hand, we have

$$[[\xi, \eta]_{\mu, \mathcal{F}}, \gamma]_{\mu, \mathcal{F}} = \mu_1(\mathcal{F}([\xi, \eta]_{\mu, \mathcal{F}}))(\gamma).$$

On the other hand,

$$[\xi, [\eta, \gamma]_{\mu, \mathcal{F}}]_{\mu, \mathcal{F}} - [\eta, [\xi, \gamma]_{\mu, \mathcal{F}}]_{\mu, \mathcal{F}} = [\mu_1(\mathcal{F}(\xi)), \mu_1(\mathcal{F}(\eta))](\gamma).$$

Since the Jacobiator is given by (28), we have

$$\mu_1(\mathcal{F}([\xi, \eta]_{\mu, \mathcal{F}})) = \mu_1([\mathcal{F}(\xi), \mathcal{F}(\eta)]).$$

Since (μ_0, μ_1) induces an isomorphism of the underlying 2-vector spaces, we have

$$\mathcal{F}([\xi, \eta]_{\mu, \mathcal{F}}) = [\mathcal{F}(\xi), \mathcal{F}(\eta)],$$

which completes the proof. ■

Finally, we find the corresponding Dirac structures for string type Lie 2-algebras. We first construct a suitable isomorphism from $\mathfrak{gl}(\mathbb{V})$ to itself, where \mathbb{V} is a 2-vector space given by (1) and $\mathfrak{gl}(\mathbb{V})$ is the strict Lie 2-algebra given by (6).

Since $\text{End}(\mathbb{V})$ is a 2-term DGLA, the Lie algebra $\text{End}_d^0(\mathbb{V})$ represents on $\text{End}^1(\mathbb{V})$ via (5). Consider the complex $(\text{Hom}(\wedge^\bullet \text{End}_d^0(\mathbb{V}), \text{End}^1(\mathbb{V})), d)$ for the Lie algebra cohomology $H^\bullet(\text{End}_d^0(\mathbb{V}), \text{End}^1(\mathbb{V}))$. For any linear map $\alpha : \text{End}_d^0(\mathbb{V}) \rightarrow \text{End}^1(\mathbb{V})$, we have the 2-cocycle

$$d\alpha : \wedge^2 \text{End}_d^0(\mathbb{V}) \rightarrow \text{End}^1(\mathbb{V}).$$

Define $\mu_2 : \wedge^2 \text{End}_d^0(\mathbb{V}) \rightarrow \text{End}_d^0(\mathbb{V}) \oplus \text{End}^1(\mathbb{V})$ by

$$\mu_2(A, B) = ([A, B], d\alpha(A, B)). \quad (30)$$

Then we have

Lemma 5.7. *If $d : V_1 \rightarrow V_0$ is zero, then with μ_2 given in (30), $\mu = (\mu_0 = \text{Id}, \mu_1 = \text{Id}, \mu_2)$ is an isomorphism from $\mathfrak{gl}(\mathbb{V})$ to itself.*

Proof. Since $d = 0$ the differential $\delta : \text{End}^1(\mathbb{V}) \rightarrow \text{End}_d^0(\mathbb{V})$ is also zero. Thus μ_2 is a bilinear natural transformation $[\cdot, \cdot] \rightarrow [\cdot, \cdot]$. The coherence condition (22) is equivalent to the fact that the second component of μ_2 is a 2-cocycle in $(\text{Hom}(\wedge^\bullet \text{End}_d^0(\mathbb{V}), \text{End}^1(\mathbb{V})), d)$. Since $d\alpha$ is (even exact) closed, (22) is automatically satisfied. ■

A quadratic Lie algebra is a Lie algebra $(V, [\cdot, \cdot]_V)$ together with a nondegenerate inner product $\langle \cdot, \cdot \rangle$ which is invariant under the adjoint action ad . A Courant algebroid over a point is exactly a quadratic Lie algebra. Recall from [16], given a quadratic Lie algebra $(V, [\cdot, \cdot]_V, \langle \cdot, \cdot \rangle)$, there is an associated 2-term L_∞ -algebra whose degree-1 part is \mathbb{R} , degree-0 part is V , differential d is zero and l_2, l_3 are given by

$$\begin{aligned} l_2(u, v) &= [u, v]_V, \quad \forall u, v \in V, \\ l_2(u, r) &= 0, \quad \forall r \in \mathbb{R}, \\ l_2(r, r') &= 0, \quad \forall r, r' \in \mathbb{R}, \\ l_3(u, v, w) &= \langle [u, v]_V, w \rangle, \quad \forall u, v, w \in V. \end{aligned}$$

We denote the corresponding Lie 2-algebra by \mathcal{V} :

$$\begin{array}{c} \mathcal{V}_1 := V \oplus \mathbb{R} \\ \mathcal{V} = \begin{array}{c} s \downarrow t \\ \mathcal{V}_0 := V. \end{array} \end{array} \quad (31)$$

The bracket functor $[\cdot, \cdot]_{\mathcal{V}}$ is given by

$$[(u, r), (v, r')]_{\mathcal{V}} = ([u, v]_V, 0),$$

and the Jacobiator J is given by

$$J_{u,v,w} = ([[u, v]_V, w]_V, l_3(u, v, w)).$$

This sort of Lie 2-algebra is called a *string-type Lie 2-algebra* in [15] for the reason that when V is a semisimple Lie algebra equipped with its Killing form (which is adjoint invariant), for example $\mathfrak{so}(n)$, we arrive at the concept of a string Lie 2-algebra (see also Example 5.9).

Denote by $\text{ad}_{\mathcal{V}} : \mathcal{V} \rightarrow \mathfrak{gl}(\mathcal{V})$ the induced linear functor by the bracket functor $[\cdot, \cdot]_{\mathcal{V}}$. Denote by $\mathfrak{G}_{\text{ad}_{\mathcal{V}}}$ the graph of the linear functor $\text{ad}_{\mathcal{V}}$.

Evidently, $\mathfrak{gl}(\mathcal{V})$ is given by

$$\begin{array}{c} \mathfrak{gl}(V) \oplus \mathbb{R} \oplus \text{End}(V, \mathbb{R}) \\ \mathfrak{gl}(\mathcal{V}) = \begin{array}{c} s \downarrow t \\ \mathfrak{gl}(V) \oplus \mathbb{R}. \end{array} \end{array} \quad (32)$$

Take any complement vector space $\text{Im}(\text{ad})^{\perp}$ of the subvector space of the image of ad in $\mathfrak{gl}(V)$. Let $\alpha : \text{End}_d^0(\mathcal{V}) = \mathfrak{gl}(V) \oplus \mathbb{R} \rightarrow \text{End}^1(\mathcal{V}) = \text{End}(V, \mathbb{R})$ be given by

$$\alpha(\text{ad}_u + X + r)(v) = \langle u, v \rangle, \quad \forall u \in V, X \in \text{Im}(\text{ad})^{\perp}, r \in \mathbb{R}. \quad (33)$$

By Lemma 5.7, $\mu = (\mu_0 = \text{Id}, \mu_1 = \text{Id}, \mu_2 = [\cdot, \cdot] + d\alpha)$ is an isomorphism from $\mathfrak{gl}(\mathcal{V})$ to itself. Since $\langle \cdot, \cdot \rangle$ is an invariant inner product on V , we have

$$\begin{aligned} d\alpha(\text{ad}_u, \text{ad}_v)(w) &= [\text{ad}_u, \alpha(\text{ad}_v)](w) - [\text{ad}_v, \alpha(\text{ad}_u)](w) - \alpha([\text{ad}_u, \text{ad}_v])(w) \\ &= -\langle v, [u, w]_V \rangle + \langle u, [v, w]_V \rangle - \langle [u, v]_V, w \rangle \\ &= \langle [u, v]_V, w \rangle. \end{aligned}$$

Therefore, comparing to the Jacobiator J of \mathcal{V} , we have

$$\begin{aligned} \mu_2(\text{ad}_u, \text{ad}_v)(w) &= ([\text{ad}_u, \text{ad}_v], d\alpha(\text{ad}_u, \text{ad}_v))(w) \\ &= (\text{ad}_{[[u, v]_V, w]}(w), d\alpha(\text{ad}_u, \text{ad}_v)(w)) \\ &= ([[u, v]_V, w]_V, \langle [u, v]_V, w \rangle) \\ &= J_{u,v,w}. \end{aligned}$$

Thus by Theorem 5.6, we have

Proposition 5.8. *Let \mathcal{V} be a string-type Lie 2-algebra as in (31) and $\mu = (\mu_0 = \text{Id}, \mu_1 = \text{Id}, \mu_2 = [\cdot, \cdot] + d\alpha)$ be an isomorphism from $\mathfrak{gl}(\mathcal{V})$ to itself with α given by (33). Then the Dirac structure $\mathfrak{G}_{\text{ad}_{\mathcal{V}}}$ of the μ -twisted omni-Lie 2-algebra $\mathfrak{gl}(\mathcal{V}) \oplus_{\mu} \mathcal{V}$ corresponds to the string-type Lie 2-algebra structure on \mathcal{V} under the correspondence of Theorem 5.6.*

In particular, we realize string Lie 2-algebras as Dirac structures of twisted omni-Lie 2-algebras.

Example 5.9. We consider the Lie algebra $\mathfrak{so}(3)$, which is isomorphic to \mathbb{R}^3 as vector spaces. Let e_1, e_2, e_3 be the basis of \mathbb{R}^3 and \cdot be the canonical inner product on \mathbb{R}^3 , then the Lie bracket is given by

$$[e_1, e_2] = \frac{1}{2}e_3, \quad [e_2, e_3] = \frac{1}{2}e_1, \quad [e_3, e_1] = \frac{1}{2}e_2.$$

The invariant inner product $\langle \cdot, \cdot \rangle$ which is given by Killing form turns out to be

$$\langle e_i, e_j \rangle = e_i \cdot e_j = \delta_j^i = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Denote the set of 3×3 symmetric matrices by $\text{Symm}(3)$. There is a canonical decomposition

$$\mathfrak{gl}(3) = \mathfrak{so}(3) \oplus \text{Symm}(3).$$

Since we have $\text{Im}(\text{ad}) = \mathfrak{so}(3)$, we can define $\alpha : \mathfrak{so}(3) \oplus \text{Symm}(3) \oplus \mathbb{R} \rightarrow \text{End}(\mathbb{R}^3, \mathbb{R})$ by

$$\alpha(\text{ad}_u, S, r)(v) = u \cdot v, \quad \forall \text{ad}_u \in \mathfrak{so}(3), \quad S \in \text{Symm}(3), \quad r \in \mathbb{R}, \quad v \in \mathbb{R}^3.$$

Let \mathcal{W} be the 2-vector space associated to the 2-term complex $\mathbb{R} \xrightarrow{0} \mathbb{R}^3$, and μ be the isomorphism $\mathfrak{gl}(\mathcal{W}) \rightarrow \mathfrak{gl}(\mathcal{W})$ given by $\mu_0 = \text{Id}, \mu_1 = \text{Id}, \mu_2 = [\cdot, \cdot] + d\alpha$. Then under the correspondence of Theorem 5.6, the string Lie 2-algebra structure on \mathcal{W} corresponds to the Dirac structure $\mathfrak{G}_{\mathcal{F}}$ of $\mathfrak{gl}(\mathcal{W}) \oplus_{\mu} \mathcal{W}$, where $\mathfrak{G}_{\mathcal{F}}$ is the graph of the functor $\mathcal{F} : \mathcal{W} \rightarrow \mathfrak{gl}(\mathcal{W})$ given by

$$\mathcal{F}(w_0 + w_1)(w'_0 + w'_1) := [w_0, w'_0]_{\mathfrak{so}(3)}, \quad \forall w_0, w'_0 \in \mathbb{R}^3, w_1, w'_1 \in \mathbb{R}.$$

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